To satisfy the nondimensional requirement of N, e_j must satisfy a set of linear, homogeneous equations

$$\sum_{j=1}^{n} a_{ij}\, e_j = 0$$

In solving these equations for e_j in a systematic manner, it is convenient to transform the dimensional matrix to a unitized form by the following operations: 1) interchanging rows, 2) multiplying a row by a nonzero constant, 3) adding one row to another, and 4) if necessary, interchanging columns.

As a result, the number of rows in the unitized form may be less than in the dimensional matrix. A unitized matrix with four rows is of the form

$$\|b_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 & 0 & \ldots & b_{1n} \\ 0 & 1 & 0 & 0 & \ldots & b_{2n} \\ 0 & 0 & 1 & 0 & \ldots & b_{3n} \\ 0 & 0 & 0 & 1 & \ldots & b_{4n} \end{Vmatrix} \qquad (3)$$

Since the foregoing row and column operations also may be applied to the equations of exponents without affecting their results, the unitized matrix may be used to form a new set of equations:

$$\sum_{j=1}^{n} b_{ij}\, e_j = 0$$

By matrix theory, the number of independent equations of such a set equals the number of nonzero rows in the unitized matrix. That number also equals the rank of the matrix. Therefore, with r nonzero rows, $e_1, \ldots e_r$ may be considered dependent, and $e_{r+1}, \ldots e_n$ may be considered independent. The exponents e_{jk} of a single similarity number N_k may be obtained by letting one of the independent exponents $e_{r+k} = 1$ and letting the other independent exponents equal zero. The result is

$$N_k = u_1^{e_{1k}}\, u_2^{e_{2k}} \ldots u_r^{e_{rk}}\, u_{r+k} \qquad (4)$$

The possibilities for k are $k = 1, \ldots (n - r)$. Therefore, a principle of dimensional analyses is that the number of similarity numbers in a set is $s = n - r$, where n and r are the number of columns and rows in the unitized dimensional matrix.

Influence of Constant Disturbing Torques on the Motion of Gravity-Gradient Stabilized Satellites

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THE conditions under which a body subject to gravitational-gradient torques will perform stable oscillations about an equilibrium point are well known.[1, 2] However, in most practical cases, torques other than those due to the gravity gradient will act on the satellite.[3] The purpose of this note is to demonstrate the effect of a constant disturbing torque upon the transient response of a gravity-gradient stabilized body.

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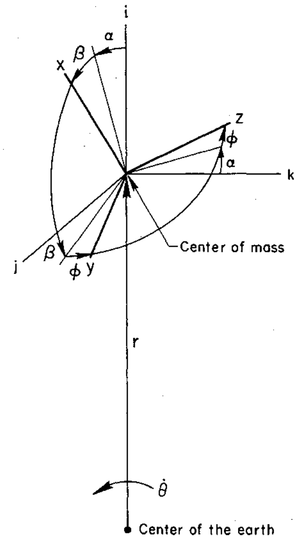

Fig. 1 Orientation of body axes with respect to orbital local horizontal coordinates

It is assumed that the satellite is on a circular orbit. Thus the oblateness and other asymmetries of the earth are neglected. The only torques that act on the body are those due to the gravity gradient and the constant disturbance.

Euler's rotational equations of motion are

$$\dot\omega_x - R_x\omega_y\omega_z = (M_x/I_x)_G + (M_x/I_x)_D \qquad (1a)$$

$$\dot\omega_y - R_y\omega_z\omega_x = (M_y/I_y)_G + (M_y/I_y)_D \qquad (1b)$$

$$\dot\omega_z - R_z\omega_x\omega_y = (M_z/I_z)_G + (M_z/I_z)_D \qquad (1c)$$

where

$$R_x = (I_y - I_z)/I_x$$

$$R_y = (I_z - I_x)/I_y$$

$$R_z = (I_x - I_y)/I_z$$

The subscript G refers to the gradient torque, whereas D indicates a disturbing torque. From the form of Eqs. (1a–1c), it can be seen that the $x\ y\ z$ axes are central principal axes.

The orientation of the body with respect to the local horizontal coordinates is defined by the angles α, β, and φ (see Fig. 1). In terms of the orientation angles and their derivatives, the body angular rates are

$$\omega_x = \dot\varphi + (\dot\alpha + \dot\theta)\sin\beta \qquad (2a)$$

$$\omega_y = \beta\sin\varphi + (\dot\alpha + \dot\theta)\cos\beta\cos\varphi \qquad (2b)$$

$$\omega_z = \beta\cos\varphi - (\dot\alpha + \dot\theta)\cos\beta\sin\varphi \qquad (2c)$$

where $\dot\theta$ is the angular rate of the local horizontal axes due to the orbital motion. Finally, the gravitational-gradient torques are[4]

$$(M_x/I_x)_G = -3\dot\theta^2 R_x(\sin\alpha\cos\varphi + \cos\alpha\sin\beta\sin\varphi) \times (\sin\alpha\sin\varphi - \cos\alpha\sin\beta\cos\varphi) \qquad (3a)$$

$$(M_y/I_y)_G = -3\dot\theta^2 R_y\cos\alpha\cos\beta(\sin\alpha\cos\varphi + \cos\alpha\sin\beta\sin\varphi) \qquad (3b)$$

$$(M_z/I_z)_G = -3\dot\theta^2 R_z\cos\alpha\cos\beta(\sin\alpha\sin\varphi - \cos\alpha\sin\beta\cos\varphi) \qquad (3c)$$

Consider the case in which there is a steady state pitch angle. Such a condition might arise physically due to residual drag forces acting in conjunction with a center-of-mass, center-of-pressure separation. Thus a steady-state value of α develops until the gradient torque is equal to the disturbance in magnitude.

If only the small disturbances about the steady-state equilibrium condition are considered, then

$$\alpha = \alpha_{ss} + \delta\alpha \quad (4a)$$

$$\beta = \delta\beta \quad (4b)$$

$$\varphi = \delta\varphi \quad (4c)$$

Linearizing Eqs. (1–3) about the equilibrium condition α_{ss} yields

$$(M_y/I_y)_D = 3\dot{\theta}^2 R_y \cos\alpha_{ss} \sin\alpha_{ss} \quad (5a)$$

$$\delta\ddot{\alpha} + 3\dot{\theta}^2 R_y\, \delta\alpha(\cos^2\alpha_{ss} - \sin^2\alpha_{ss}) = 0 \quad (5b)$$

$$\delta\ddot{\beta} - \delta\beta(R_z\dot{\theta}^2 + 3R_z\dot{\theta}^2 \cos^2\alpha_{ss}) - \delta\dot{\varphi}(1 + R_z)\dot{\theta} + 3\delta\varphi R_z\, \dot{\theta}^2 \sin\alpha_{ss} \cos\alpha_{ss} = 0 \quad (5c)$$

$$\delta\ddot{\varphi} + \delta\varphi(R_x\dot{\theta}^2 + 3R_x\dot{\theta}^2 \sin^2\alpha_{ss}) + \delta\dot{\beta}(1 - R_x)\dot{\theta} - 3\delta\beta R_x\dot{\theta}^2 \sin\alpha_{ss} \cos\alpha_{ss} = 0 \quad (5d)$$

In the absence of a disturbing torque, R_x and R_y are normally positive and R_z negative for stable behavior.† Thus

$$I_y \geq I_z > I_x \quad (6)$$

When a disturbing torque is present, the pitch response essentially is unaltered if α_{ss} is small. However, Eqs. (5c) and (5d) now have a $\delta\varphi$ and a $\delta\beta$ term, respectively, if α_{ss} is different from zero. The characteristic equation for the coupled roll-yaw motion is

$$\lambda^4 + \lambda^2\dot{\theta}^2(1 + 3R_x \sin^2\alpha_{ss} - 3R_z \cos^2\alpha_{ss} - R_zR_x) - 3\lambda\dot{\theta}^3 \sin\alpha_{ss} \cos\alpha_{ss}(R_x + R_z) - 4R_xR_z\dot{\theta}^4 = 0 \quad (7)$$

If α_{ss} is small, the four roots are, approximately,

$$\lambda_1 = \dot{\theta}[-a + \tfrac{1}{2}(b)^{1/2}]^{1/2} + [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}] \sin\alpha_{ss} \quad (8a)$$

$$\lambda_2 = -\dot{\theta}[-a + \tfrac{1}{2}(b)^{1/2}]^{1/2} + [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}] \sin\alpha_{ss}$$

$$\lambda_3 = \dot{\theta}[-a - \tfrac{1}{2}(b)^{1/2}]^{1/2} - [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}] \sin\alpha_{ss} \quad (8b)$$

$$\lambda_4 = -\dot{\theta}[-a - \tfrac{1}{2}(b)^{1/2}]^{1/2} - [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}] \sin\alpha_{ss}$$

where

$$a = (\tfrac{1}{2})(1 - 3R_z - R_zR_x)$$
$$b = (1 - 3R_z - R_zR_x)^2 + 16\, R_zR_x$$

For the moment-of-inertia distribution given by Eq. (6), λ_1 and λ_2 are complex conjugates with negative real parts, whereas λ_3 and λ_4 are conjugates with positive real parts.‡ Thus the roll-yaw motion is unstable. If the sum of R_x and R_z is zero, the real parts of Eqs. (8a) and (8b) are zero, but the pitch restoring torque also is zero! Digital solutions of the nonlinear differential equations have verified this instability.

A disturbing torque that causes a steady-state value of β also is possible. In this case the three perturbation equations are coupled, and the characteristic equation is of the sixth degree. For small values of β_{ss}, the roots do not have any positive real parts and the solutions are stable. However, the digital solutions of the nonlinear equations diverged for values of β_{ss} greater than about 10°.

An examination of Eq. (3a) indicates that a constant torque about the x axis requires that at least two of the three orientation angles have steady-state values. This case has not been examined extensively, but a limited number of digital solutions were stable for small values of $(M_x)_D$.

The fact that a bias angle in pitch leads to instability in the roll-yaw motion indicates the highly nonlinear nature of the problem. A similar roll-yaw behavior is caused by the forced pitch motion due to orbital eccentricity.[5] Thus it is apparent that the principle of superposition is of limited use in stability analyses of gravity-gradient stabilized satellites.

† Another conditionally stable configuration is discussed in Ref. 2.

‡ This assumes α_{ss} to be positive.

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Entropy Perturbations in One-Dimensional Magnetohydrodynamic Flow

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IN a method developed by Germain and the author[1–3] for discussing weakly nonisentropic one-dimensional flows of an ideal compressible fluid, it was found that the addition of an entropy perturbation introduced a nonhomogeneous term in an otherwise homogeneous system of perturbation equations, and, further, that the entropy perturbation could be determined directly. Thus, the various problems considered could be solved by considering first the homogeneous system (isentropic perturbed flow) and then adding particular solutions to the complete system (nonisentropic perturbed flow). In two cases of interest, viz., an initially uniform or centered simple-wave flow, it was found that the addition of an entropy perturbation affected the sound speed but not the particle velocity, i.e., there was a *particular* solution with the particle-velocity perturbation equal to zero. A general discussion of this phenomenon, including necessary and sufficient conditions for it to occur, was given in Refs. 2 and 3.

The forementioned perturbation theory has been extended to one-dimensional hydromagnetic flow subjected to a transverse magnetic field,[4–7] and it was found that the addition of an entropy perturbation did not affect the particle velocity in an initially uniform flow. (This is a consequence of the result that the nonisentropic perturbation of an initially uniform flow must reduce to the solution of the corresponding problem in conventional gas dynamics in the limit of vanishing magnetic field.)[8] But this result *was not* obtained for an initially centered simple-wave flow.[5] It is the purpose of the present paper to derive conditions for the particle velocity to be unaffected by the addition of an en-

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